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A NOTE ON SOLITARY AND CNOIDAL WAVES WITH SURFACE
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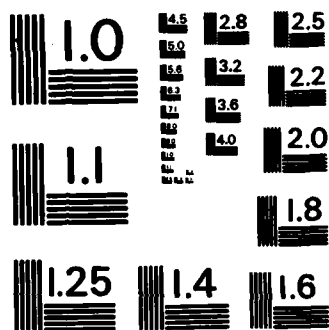
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A NOTE ON SOLITARY AND CNOIDAL
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ABSTRACT

The influence of surface tension on solitary and cnoidal waves is discussed. This problem was first considered by Korteweg and de Vries [1]. Shinbrot [2] attempted to derive Korteweg and de Vries results by a formal perturbation expansion. However a part of his results is incorrect. In this note an approach different from those in [1] and [2] is used to construct a correct perturbation solution.

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SIGNIFICANCE AND EXPLANATION

In this note we present a perturbation solution for gravity capillary waves in water of finite depth. The calculation is valid when both the amplitude of the waves and the depth of the water are small. This problem was first considered by Korteweg and de Vries [1]. These authors showed that waves of small amplitude in relatively shallow water can be described approximately by a nonlinear differential equation. In addition Korteweg and de Vries obtained periodic solutions of this equation in closed form. They named these solutions cnoidal waves. Solitary waves are the limit of cnoidal waves as the wavelength tends to infinity.

Shinbrot [2] attempted to derive Korteweg and de Vries results by a formal perturbation expansion. However, a part of his results is incorrect, because the possibility of a depression solitary wave was excluded by him.

In this note an approach different from those in [1] and [2] is used to construct a perturbation solution which allows depression waves. The results obtained agree with those presented by Korteweg and de Vries [2].

A NOTE ON SOLITARY AND CNOIDAL WAVES WITH SURFACE TENSION

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§1. INTRODUCTION

In the present work we discuss the influence of surface tension on solitary and cnoidal waves. Solitary waves with surface tension were first considered by Korteweg and de Vries [1]. They showed that depression solitary waves exist for sufficiently large values of the surface tension. Shinbrot [2] attempted to study solitary waves with surface tension by a formal perturbation expansion. However, a part of his results is incorrect, because the possibility of a depression wave was excluded by him. In this note we use an approach different from those in [1],[2] to construct a perturbation solution which allows depression waves. In addition, the effect of surface tension on cnoidal waves is also considered. The results obtained agree with those presented by Korteweg and de Vries [1].

The problem is formulated in Section 2. The perturbation expansion for solitary waves is derived in Section 3. Cnoidal waves with surface tension are considered in Section 4.

§2. FORMULATION OF THE PROBLEM

The governing equations for the steady flow of an incompressible inviscid fluid of constant density with surface tension in reference to a moving coordinate system at a constant speed c in the x -direction are the following:

$$u_x^* + v_y^* = 0, \quad (1)$$

$$\rho(u^*u_x^* + v^*u_y^*) = -p_x^*, \quad (2)$$

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$$\rho(u^*v_x^* + v^*v_y^*) = -p_y^* - \rho g, \quad (3)$$

subject to the boundary conditions

$$\zeta_\xi^* + u^*\zeta_x^* - v^* = 0 \quad \text{at } y = \zeta^*(x) \quad (4)$$

$$p^* = -T^*\zeta_{xx}^*(1 + \zeta_x^{*2})^{3/2} \quad \text{at } y = \zeta^*(x) \quad (5)$$

$$v^* = 0 \quad \text{at } y = 0. \quad (6)$$

Here (u^*, v^*) is the velocity, ρ the constant density, p the pressure, g the constant gravitational acceleration, $y = \zeta^*$ the equation of the free surface and T^* the constant surface tension coefficient. Following Peters and Stoker ([3]), we use x and the stream function ψ as independent variables, and transform the fluid domain to a fixed horizontal strip in the x, ψ -plane. Furthermore, it is also more convenient to use u^*, p^* and the so-called stream line function f^* as dependent variables, where f^* satisfies $\psi(x, f^*) = \text{constant}$ to define a stream line. If we introduce the nondimensional variables

$$\xi = x/h, \quad \eta = \psi/ch, \quad u = u^*/c, \quad v = v^*/c, \\ f = f^*/h, \quad p = p^*/\rho c^2, \quad \lambda = gh/c^2, \quad T = T^*/\rho gh^2.$$

where h is the equilibrium fluid depth, (1) to (6) become

$$u_\xi = -f_\eta p_\xi + f_\xi p_\eta, \quad -\infty < \xi < \infty, \quad (7)$$

$$uf_{\xi\xi} + u_\xi f_\xi = -\lambda f_\eta - p_\eta, \quad 0 < \eta < 1, \quad (8)$$

$$uf_\eta = 1, \quad (9)$$

$$f = 0, \quad \text{at } \eta = 0, \quad -\infty < \xi < \infty, \quad (10)$$

$$p = -Tf_{\xi\xi}/(1 + f_\xi^2)^{3/2} \quad \text{at } \eta = 1, \quad -\infty < \xi < \infty. \quad (11)$$

Assume λ is close to some critical value ℓ and let

$$\varepsilon = \ell - \lambda, \quad \sigma = \xi(\varepsilon)^{1/2}, \quad (12)$$

then (7) to (11) in terms of σ become

$$u_\sigma = -f_\eta p_\sigma + f_\sigma p_\eta, \quad (13)$$

$$\varepsilon(uf_{\sigma\sigma} + u_{\sigma}f_{\sigma}) = \varepsilon f_{\eta} - lf_{\eta} - p_{\eta} \quad (14)$$

$$uf_{\eta} = 1, \quad (15)$$

$$f = 0, \text{ at } \eta = 0, \quad (16)$$

$$p = -\tau \varepsilon f_{\sigma\sigma} / (1 + \varepsilon f_{\sigma}^2)^{3/2} \text{ at } \eta = 1. \quad (17)$$

§3. SOLITARY WAVE SOLUTIONS

To solve (13) to (17), we assume

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \quad (18)$$

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \dots \quad (19)$$

$$p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots \quad (20)$$

where

$$u_0 = 1, \quad f_0 = \eta, \quad p_0 = -l(\eta - 1). \quad (21)$$

Substitution of (18) to (21) in (13) to (17) will yield a sequence of equations and boundary conditions for the successive approximations. The equations for the first approximations are

$$u_{1\sigma} = -p_{1\sigma} - lf_{1\sigma}, \quad (22)$$

$$1 - lf_{1\eta} - p_{1\eta} = 0, \quad (23)$$

$$u_1 + f_{1\eta} = 0, \quad (24)$$

$$f_1(\sigma, 0) = 0, \quad (25)$$

$$p_1(\sigma, 0) = 0. \quad (26)$$

From (21) to (22), we have

$$f_{1\sigma\eta\eta} = 0. \quad (27)$$

By integration and making use of (25), we obtain

$$f_{1\sigma} = a_1'(\sigma)\eta,$$

where $a_1(\sigma)$ is to be determined. By integration again,

$$f_1 = a_1(\sigma)\eta + b_1(\eta) .$$

Here we assume that $a_1(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$ or $a_1(\sigma)$ is a periodic function with period L and $\int_0^L a_1(\sigma) d\sigma = 0$. It follows that

$$f_1 = a_1(\sigma)\eta . \quad (28)$$

From (23), (24) and (26), we have

$$u_1 = -a_1(\sigma) , \quad (29)$$

$$p_1 = [1 - la_1(\sigma)](\eta - 1) . \quad (30)$$

By substituting (28) to (30) in (22), it is found that

$$a_1'(\sigma)(l - 1) = 0 . \quad (31)$$

If $a_1'(\sigma) \neq 0$, then

$$l = 1 . \quad (32)$$

The equations for the second approximations are

$$u_{2\sigma} = -f_{2\sigma} - p_{2\sigma} + a_1'(\sigma)\eta - a_1(\sigma)a_1'(\sigma) , \quad (33)$$

$$a_1''(\sigma)\eta = a_1(\sigma) - f_{2\eta} - p_{2\eta} , \quad (34)$$

$$f_{2\eta} + u_2 - a_1^2(\sigma) = 0 , \quad (35)$$

$$f_2(\sigma, 0) = 0 , \quad (36)$$

$$p_2 = -Ta_1''(\sigma) \text{ at } \eta = 1 , \quad (37)$$

where (28) to (30) and (32) have been used. As before we find that

$$f_2 = a_2(\sigma)\eta - a_1''(\sigma)\eta^3/6 , \quad (38)$$

$$u_2 = -a_2(\sigma) + a_1^2(\sigma) + a_1''(\sigma)\eta^2/2 , \quad (39)$$

$$p_2 = -a_2(\sigma)(\eta - 1) + a_1(\sigma)(\eta - 1) + a_1'(\sigma)(-T + \eta^3/6 - \eta^2/2 + 1/3) . \quad (40)$$

Substitution of (38) and (40) in (33) yields

$$(1/3 - T)a_1''(\sigma) = a_1'(\sigma) - 3a_1(\sigma)a_1'(\sigma) . \quad (41)$$

If we impose the condition $a_1'(\sigma), a_1''(\sigma) \rightarrow 0$ as $\sigma \rightarrow \infty$ and $a_1'(0) = 0$, then by integration we obtain

$$a_1(\sigma) = \text{sech}^2(\sigma/2)(1/3 - T)^{-1/2} \quad (42)$$

It follows from (19), (21), (28) and (42) that

$$f^* \sim h + h(1 - \lambda)\text{sech}^2(x/2h)[(1 - \lambda)/(1/3 - T)]^{1/2}. \quad (43)$$

If $T < 1/3$ and $\lambda < 1$, there is a solitary wave of elevation. In this case, $c^2 > gh$ and the solitary wave moves at a supercritical speed. If

$T > 1/3$ and $\lambda > 1$, there is a solitary wave of depression moving at a subcritical speed units $c^2 < gh$. We may define

$$a = |h(1 - \lambda)|,$$

and (43) in terms of a becomes

$$f^* \sim h \pm a \text{sech}^2(x/2h)[\pm a/(1/3 - T)]^{1/2}. \quad (44)$$

If we consider a periodic solution of (41), then by integration

$$(1/3 - T)[a_1'(\sigma)]^2 = -a_1^3(\sigma) + a_1^2(\sigma) + c_1 a_1(\sigma) + c_2, \quad (45)$$

where c_1, c_2 are two arbitrary constants. The periodic solutions are called cnoidal waves and will be discussed in the next section. Finally if

$T = 1/3$, then (41) becomes

$$a_1'(\sigma)(1 - 3a_1(\sigma)) = 0, \quad (46)$$

and the only possible solution is $a_1(\sigma) = 0$.

§4. CNOIDAL WAVE SOLUTION

For a bounded, periodic solution of (45), it is necessary that the right hand side of (45) must have three real simple zeros, and the solution will oscillate periodically between two of the zeros. Therefore, we rewrite (45) in the form

$$(1/3 - T)[a_1'(\sigma)]^2 = [h_1 - a_1(\sigma)][a_1(\sigma) - h_2][a_1(\sigma) - \gamma], \quad (47)$$

where h_1, h_2 are the extremum values of $a_1(\sigma)$ and

$$\gamma = 1 - h_2 - h_1. \quad (48)$$

Without loss of generality we shall assume that h_1 and h_2 satisfy the inequality

$$\left(\frac{1}{3} - T\right)(1 - \lambda)(h_1 - h_2) > 0$$

Following Korteweg and de Vries [1] we introduce the parameter X and write

$$a_1(\sigma) = h_1 \cos^2 X + h_2 \sin^2 X. \quad (49)$$

Substituting (49) into (47), we obtain

$$\beta \frac{dX}{d\sigma} = (1 - K^2 \sin^2 X)^{1/2}. \quad (50)$$

Here β and K^2 are defined by

$$\beta = \left(\frac{4}{3} \frac{1 - 3T}{h_1 - \gamma}\right)^{1/2}, \quad (51)$$

$$K^2 = \frac{h_1 - h_2}{h_1 - \gamma}. \quad (52)$$

We take the origin of σ at a point where $a_1(\sigma)$ attains the value h_1 . Thus integrating (50) we obtain

$$\sigma = \beta \int_0^X \frac{dX}{(1 - K^2 \sin^2 X)^{1/2}} = \beta F(X, K). \quad (53)$$

We can rewrite (53) in terms of Jacobian elliptic functions as

$$\cos X = \operatorname{cn} \frac{\sigma}{\beta}. \quad (54)$$

Substituting (56) into (49) we have

$$a_1(\sigma) = h_2 + (h_1 - h_2) \operatorname{cn}^2 \frac{\sigma}{\beta}. \quad (55)$$

It follows from (19), (21), (28) and (55) that

$$f^* \sim h + (1-\lambda)hh_2 + (1-\lambda)h(h_1 - h_2) \operatorname{cn}^2\left(\frac{\sigma}{\beta}\right) [(h_1 - \gamma)(1-\lambda)/(1/3 - T)]^{1/2} \quad (56)$$

The wavelength L of the wave is given by

$$L = 4h \left[\frac{1 - 3T}{3(h_1 - \gamma)(1 - \lambda)} \right]^{1/2} F_1(K) , \quad (57)$$

where

$$F_1(K) = \int_0^{\pi/2} (1 - K^2 \sin^2 x)^{-1/2} dx . \quad (58)$$

The determination of the solution is completed by imposing the condition $\int_0^L a_1(\sigma) d\sigma = 0$ where the integral is over a wavelength. Using (49), (50) and (52) we have

$$\begin{aligned} 0 &= \beta \int_0^{\pi/2} \frac{h_1 \cos^2 x + h_2 \sin^2 x}{(1 - K^2 \sin^2 x)^{1/2}} dx \\ &= \beta [\gamma F_1(K) + (h_1 - \gamma) E_1(K)] . \end{aligned} \quad (59)$$

Here

$$E_1(K) = \int_0^{\pi/2} (1 - K^2 \sin^2 x)^{1/2} dx .$$

Hence

$$\gamma F_1(K) + (h_1 - \gamma) E_1(K) = 0 . \quad (60)$$

The eight quantities λ , h_1 , h_2 , γ , K , L , T and β are related by the five equations (48), (51), (52), (57) and (60). If for instance λ , L and T are given the others can be found.

If $L = \infty$, relations (57), (52) (60) and (48) yield $K = 1$, $\gamma = h_2 = 0$ and $h_1 = 1$. The cnoidal wave reduces then to the solitary wave considered in Section 3.

For infinitesimal amplitude, equation (41) reduces to

$$(1/3 - T) a_1''(\sigma) = a_1'(\sigma) . \quad (61)$$

The solution is

$$a_1(\sigma) = h_1 \cos(T - 1/3)^{-1/2} \sigma \quad (62)$$

It follows from (19), (21), (28) and (62) that

$$f^* = h + h_1(1 - \lambda) \cos \frac{x}{h} [(1 - \lambda)/(T - 1/3)]^{1/2} . \quad (63)$$

If we introduce

$$\kappa = [(1 - \lambda)/(T - 1/3)]^{1/2}/h , \quad (64)$$

we have

$$f^* = h + h h_1(1 - \lambda) \cos \kappa x . \quad (65)$$

Relation (64) can be rewritten as

$$\frac{gh}{c^2} = 1 - \kappa^2 h^2 (T - 1/3) . \quad (66)$$

The dispersion relation of linear sine waves is given by

$$c^2 = \frac{g}{\kappa} \tanh \kappa h [1 + \frac{T^*}{\rho g} \kappa^2] . \quad (67)$$

In the limit as $\kappa h \rightarrow 0$, (67) may be expanded as

$$\frac{c^2}{gh} \sim 1 + (T - 1/3) \kappa^2 h^2 + o(\kappa^2 h^2) . \quad (68)$$

A comparison of (67) and (68) indicates that the cnoidal wave solution overlaps the classical linear theory in the limit as the amplitude of the waves tends to zero.

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